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DISTRIBUTIONAL WATSON TRANSFORMS*

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Abstract. For all Watson transforms W in $L^2(\mathbb{R}_+)$ a triple of Hilbert space $\mathscr{L}_G \subset L^2(\mathbb{R}_+) \subset \mathscr{L}'_G$ is constructed such that W may be extended to $\mathscr{L}'_G$. These results allow the construction of a triple $\mathscr{L} \subset L^2(\mathbb{R}_+) \subset \mathscr{L}'$, where $\mathscr{L}$ is a Gelfand–Fréchet space. This leads to a theory of distributional Watson transforms.

Introduction. Guinand [2], [3], Miller [5], [6], [7] and Goldberg [1] have considered linear manifolds, dense in $L^2(\mathbb{R}_+)$, which are invariant under Watson transforms and equipped with a suitable inner product form a Hilbert space. These manifolds were generalized in [11].

Miller [7], [8], [9] has constructed the dual of such linear manifolds, which is of course a Hilbert space, containing a copy of $L^2(\mathbb{R}_+)$. By means of this dual, Miller [7], [8] extended Watson transforms beyond $L^2(\mathbb{R}_+)$.

In this note we wish to show that the results in [11] make it possible to construct a linear manifold in $L^2(\mathbb{R}_+)$, which is invariant under Watson transforms; however, this linear manifold is not a Hilbert space, but a Gelfand–Fréchet space. This leads to a theory of distributions and Watson transforms defined on them. For all Watson transforms we may choose the same Gelfand–Fréchet space. Specialization of the Gelfand–Fréchet space in $L^2(\mathbb{R}_+)$ leads to a theory analogous to the distributional Fourier transform on the space of tempered distributions.

1. Preliminaries. For $K \in L^\infty(\mathbb{R})$ the linear operator $M[K]$ on $L^2(\mathbb{R})$ is defined by $M[K]f = Kf$. The operator P on $L^2(\mathbb{R}_+)$ is given by $(Pf)(x) = (1/x)f(1/x)$. By $\mathscr{M}$ we denote the Mellin transform, which is an isometry from $L^2(\mathbb{R}_+)$ onto $L^2(\mathbb{R})$, cf. [13]. In [10] it was shown that every Watson transform on $L^2(\mathbb{R}_+)$ can be written as

$$W = \mathscr{M}^{-1}M[K]\mathscr{M}P, \qquad K \in L^\infty(\mathbb{R}). \tag{1.1}$$

To denote the dependence on K we shall also write W_K instead of W. In [11] the operator $V\,(=V_G)$ on $L^2(\mathbb{R}_+)$ was introduced by

$$V = \mathscr{M}^{-1}M[G]\mathscr{M}, \qquad G \in L^\infty(\mathbb{R}).$$

In the sequel we shall consider a sequence of functions $G_i \in L^\infty(\mathbb{R})$, $i = 1, 2, \cdots$. We therefore introduce the following notation:

$$V_i = V_{G_i},$$
$$\overline{V}_i = V_{\bar{G}_i},$$
$$V_{ji} = V_{G_j/G_i} \quad \text{if } G_j/G_i \in L^\infty(\mathbb{R}),$$

and

$$V'_G = V_{G'},$$

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where

$$G'(x) = G(-x).$$

In [11] it was shown that $V'W = WV$ and hence W maps L_G into $L_{G'}$, where L_G and $L_{G'}$ are the ranges of V_G and $V_{G'}$ respectively.

If $G \neq 0$ a.e., then $V\,(=V_G)$ is injective and equipped with the inner product

$$(Vf, Vg)_{L_G} = (f, g)_{L^2(\mathbb{R}_+)}, \qquad f, g \in L^2(\mathbb{R}_+).$$

L_G then is a Hilbert space, isometrically isomorphic to $L^2(\mathbb{R}_+)$. In this case W maps L_G continuously into $L_{G'}$. Instead of the notation $(\cdot,\cdot)_{L_G}$, we shall also use $(\cdot,\cdot)_G$.

PROPOSITION 1. *Let $G_1, G_2 \in L^\infty(\mathbb{R})$ satisfy $G_i \neq 0$ a.e., $i = 1, 2$, and $G_2/G_1 \in L^\infty(\mathbb{R})$. Then*

$$L_{G_2} \subset L_{G_1}, \tag{1.2}$$

(1.3) *the injection $I: L_{G_2} \to L_{G_1}$ is continuous and has a dense range.*

Proof. If $g \in L_{G_2}$, then for some $f \in L^2(\mathbb{R}_+)$, $g = V_2 f$ and $g = V_1 h$ with $h = V_{21} f \in L^2(\mathbb{R}_+)$. Hence (1.2) holds. Using the same notation, we obtain

$$\begin{aligned}\|g\|_{G_1} = \|h\|_{L^2(\mathbb{R}_+)} = \left\|\frac{G_2}{G_1}\mathscr{M}f\right\|_{L^2(\mathbb{R})} &\leqq \left\|\frac{G_2}{G_1}\right\|_{L^\infty(\mathbb{R})} \|f\|_{L^2(\mathbb{R}_+)} \\ &= \left\|\frac{G_2}{G_1}\right\|_{L^\infty(\mathbb{R})} \cdot \|g\|_{G_2}.\end{aligned}$$

This implies the continuity of I in (1.3). The remainder of (1.3) follows from

$$(V_1 u, g)_{G_1} = (u, h)_{L^2(\mathbb{R}_+)} = (\mathscr{M}u, (G_2/G_1)\mathscr{M}f)_{L^2(\mathbb{R})}, \quad u \in L^2(\mathbb{R}_+), \quad g \in L_{G_2}.$$

Since $M[G_2/G_1]$ has a dense range in $L^2(\mathbb{R})$, it follows that if the left-hand side equals zero for fixed $u \in L^2(\mathbb{R}_+)$ and arbitrary $g \in L_{G_2}$, then the right-hand side shows that $\mathscr{M}u = 0$ and thus $V_1 u = 0$. This completes the proof.

For $G \in L^\infty(\mathbb{R})$, $G \neq 0$ a.e., denote by L_G' the dual of L_G.

PROPOSITION 2. *Let $G \in L^\infty(\mathbb{R})$ satisfy $G \neq 0$ a.e. Then L_G' is (isomorphic to) the completion of $L^2(\mathbb{R}_+)$ under the norm $\|\overline{V}f\|_{L^2(\mathbb{R}_+)}$.*

Proof. Denote by A and B the antilinear isometries from L_G onto L_G' and $L^2(\mathbb{R}_+)$ onto $L^{2\prime}(\mathbb{R}_+)$ respectively. Then the restriction of Bf to L_G belongs to L_G' and $A^{-1}Bf = V\overline{V}f, f \in L^2(\mathbb{R}_+)$, for

$$|(Bf)(Vg)| = |(Vg, f)_{L^2(\mathbb{R}_+)}| = |(g, \overline{V}f)_{L^2(\mathbb{R}_+)}| \leqq \|\overline{V}f\|_{L^2(\mathbb{R}_+)} \|Vg\|_G$$

and

$$(Vg, A^{-1}Bf)_G = (Bf)(Vg) = (Vg, f)_{L^2(\mathbb{R}_+)} = (Vg, V\overline{V}f)_G,$$

where $g \in L^2(\mathbb{R}_+)$.

If $w \in L_G'$ and $A^{-1}w = Vh$, $h \in L^2(\mathbb{R}_+)$, then with $f \in L^2(\mathbb{R}_+)$,

$$(Bf, w)_{L_G'} = (A^{-1}Bf, A^{-1}w)_G = (V\overline{V}f, Vh)_G = (\overline{V}f, h)_{L^2(\mathbb{R}_+)}.$$

From this it follows that $BL^2(\mathbb{R}_+)$ is dense in L'_G. Also,

$$(Bf, Bf)_{L'_G} = (\bar{V}f, \bar{V}f)_{L^2(\mathbb{R}_+)}.$$

If we identify Bf with f in $L^2(\mathbb{R}_+)$, the theorem follows.

2. The space $\mathscr{L}$ and its dual $\mathscr{L}'$. Let $(G_j)_{j=1}^{\infty}$ be a sequence of functions on $\mathbb{R}$ such that

$$G_j \in L^\infty(\mathbb{R}), \qquad j = 1, 2, \cdots, \tag{2.1}$$

$$G_j \neq 0 \quad \text{a.e.}, \qquad j = 1, 2, \cdots, \tag{2.2}$$

$$G_{j+1}/G_j \in L^\infty(\mathbb{R}), \qquad j = 1, 2, \cdots, \tag{2.3}$$

and

(2.4) G_k/G_j is locally bounded on $\mathbb{R}$, $k = 1, 2, \cdots, j-1; j = 2, 3, \cdots$.

The conditions (2.1), (2.2) and (2.3) imply that we have obtained a descending chain of Hilbert spaces:

$$L^2(\mathbb{R}_+) \supset L_{G_1} \supset \cdots \supset L_{G_j} \supset L_{G_{j+1}} \supset \cdots.$$

Put $\mathscr{L} = \bigcap_{j=1}^{\infty} L_{G_j}$.

PROPOSITION 3. *$\mathscr{L}$ is dense in all L_{G_k}, $k = 1, 2, \cdots$.*

Proof. Denote by $C_0(\mathbb{R})$ the space of all continuous functions on $\mathbb{R}$ having a compact support. For each $\phi \in C_0(\mathbb{R})$, let $u_\phi = G_k\phi$. Then

$$u_\phi = G_j(G_k/G_j)\phi, \qquad j = 1, 2, \cdots.$$

By (2.3) and (2.4), $(G_k/G_j)\phi \in L^2(\mathbb{R})$, $j = 1, 2, \cdots$. Thus

$$\mathscr{M}^{-1}u_\phi = V_j\mathscr{M}^{-1}(G_k/G_j)\phi, \qquad j = 1, 2, \cdots.$$

Consequently, $\mathscr{M}^{-1}u_\phi \in \mathscr{L}$. Since for all $g \in L^2(\mathbb{R}_+)$ and $\phi \in C_0(\mathbb{R})$,

$$(\mathscr{M}^{-1}u_\phi, V_k g)_{G_k} = (\phi, \mathscr{M}g)_{L^2(\mathbb{R})}, \tag{2.5}$$

and since $C_0(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, it follows that if the left-hand side of (2.5) equals zero for a fixed $g \in L^2(\mathbb{R}_+)$ and arbitrary $\phi \in C_0(\mathbb{R})$, then the right-hand side of (2.5) implies that $\mathscr{M}g = 0$ and consequently $V_k g = 0$. Hence $\mathscr{L}$ is dense in L_{G_k}, $k = 1, 2, \cdots$.

PROPOSITION 4. *$\mathscr{L}$ is dense in $L^2(\mathbb{R}_+)$.*

Proof. We use the notation of the proof of Proposition 3. Proposition 4 follows from

$$(f, \mathscr{M}^{-1}u_\phi)_{L^2(\mathbb{R}_+)} = (\bar{G}_k\mathscr{M}f, \phi)_{L^2(\mathbb{R}_+)}, \qquad f \in L^2(\mathbb{R}_+).$$

We equip $\mathscr{L}$ with the initial topology, that is the coarsest topology such that all identity mappings $\mathscr{L} \to L_{G_j}$ are continuous. A consequence of Proposition 3 is that $\mathscr{L}$ is complete. Hence $\mathscr{L}$ is a Fréchet–Gelfand space. The dual $\mathscr{L}'$ of $\mathscr{L}$ is given by

$$\mathscr{L}' = \bigcup_{j=1}^{\infty} L'_{G_j}.$$

Here equality means that if $f \in L'_{G_j}$, then f restricted to $\mathscr{L}$ belongs to $\mathscr{L}'$ and, conversely, that if $g \in \mathscr{L}'$, then for some integer j, g can be extended to the whole space L_{G_j} and thus belongs to L'_{G_j}.

We equip $\mathscr{L}'$ with the strong topology $\beta(\mathscr{L}', \mathscr{L})$. Since $\mathscr{L}$ is bornological, $\mathscr{L}'$ is complete (see [4, p. 223]).

3. The distributional Watson transform. Let

$$\mathscr{H} = \bigcap_{j=1}^{\infty} L_{G'_j},$$

equip $\mathscr{H}$ with the initial topology, and give the dual $\mathscr{H}'$ of $\mathscr{H}$ the strong topology.

The Watson transform $W = W_K$ given by (1.1) is a mapping from $\mathscr{L}$ into $\mathscr{H}$. For, if $g \in \mathscr{L}$, then there exists a sequence $(f_j)_{j=1}^{\infty}$ in $L^2(\mathbb{R}_+)$, such that $g = V_j f_j$ and $Wg = WV_j f_j = V'_j W f_j$. Consequently, $Wg \in \mathscr{H}$. Furthermore, $W: \mathscr{L} \to \mathscr{H}$ is continuous. For, with the same notation,

$$\|Wg\|_{G'_j} = \|Wf_j\|_{L^2(\mathbb{R}_+)} \leqq \|W\|_{L^2(\mathbb{R}_+)} \|f_j\|_{L^2(\mathbb{R}_+)} = \|W\|_{L^2(\mathbb{R}_+)} \|g\|_{G_j}.$$

The adjoint W^* of W is a Watson transform given by

$$W^* = \mathscr{M}^{-1} M[\bar{K}'] \mathscr{M} P.$$

Let $\mathscr{W} = {}^t W^*$ be the transpose of W^*. It maps $\mathscr{H}'$ continuously into $\mathscr{L}'$ (see [4, p. 256]). We call $\mathscr{W}$ the distributional Watson transform. The following proposition says why.

PROPOSITION 5. $\mathscr{W}|_{L^2(\mathbb{R}_+)} = W$.

Proof. Let B be the antilinear isometry from $L^2(\mathbb{R}_+)$ onto its dual. Then $Bf \in \mathscr{H}', f \in L^2(\mathbb{R}_+)$. For each $g \in \mathscr{L}$ we have

$$\langle {}^t W^* Bf, g \rangle = \langle Bf, W^* g \rangle = (W^* g, f)_{L^2(\mathbb{R}_+)} = (g, Wf)_{L^2(\mathbb{R}_+)} = \langle BWf, g \rangle.$$

Hence ${}^t W^* B = BW$. If we identify the elements of $L^2(\mathbb{R}_+)$ and $BL^2(\mathbb{R}_+)$ the proposition follows.

Remark 1. If with $K \in L^\infty(\mathbb{R})$, also $1/K \in L^\infty(\mathbb{R})$, then W_K maps $\mathscr{L}$ onto $\mathscr{H}$ and hence $\mathscr{W}$ maps $\mathscr{H}'$ onto $\mathscr{L}'$.

Remark 2. If $K(x)K(-x) = 1$ and if $L_{G_j} = L_{G'_j}$ for all $j = 1, 2, \cdots$, then W_K is involutory and hence $\mathscr{W}$ is involutory.

4. Remarks. Let ϕ be a measurable function on $\mathbb{R}_+$ for which the integral

$$\int_0^\infty t^{-1/2} |\phi(t)| \, dt$$

is finite. Let G be defined by

$$G(x) = \int_0^\infty t^{-1/2+ix} \phi(t) \, dt, \qquad x \in \mathbb{R}.$$

Then $G \in L^\infty(\mathbb{R})$ and

$$(V_G f)(x) = \int_0^\infty \phi(xt^{-1}) t^{-1} f(t) \, dt$$

(see [12]).

For each integer j, we choose ϕ to be the function ϕ_j defined by

$$\phi_j(t) = \begin{cases} \dfrac{1}{\Gamma(j)}(1-t)^{j-1}, & 0 < t < 1, \\ 0, & t \geqq 1. \end{cases}$$

Then

$$G_j(x) = \frac{\Gamma(\frac{1}{2} + ix)}{\Gamma(\frac{1}{2} + ix + j)}, \qquad x \in \mathbb{R},$$

and the sequence $(G_j)_{j=1}^{\infty}$ satisfies the conditions (2.1), (2.2), (2.3) and (2.4). Furthermore we have

$$L_{G_j} = \left\{ f \in L^2(\mathbb{R}_+) | f(x) = \frac{1}{\Gamma(j)} \int_x^{\infty} (t-x)^{j-1} f^{(j)}(t)\, dt \right.$$
$$\left. \text{for some function } f^{(j)}(t) \text{ defined on } \mathbb{R}_+ \text{ with } t^j f^{(j)}(t) \in L^2(\mathbb{R}_+) \right\}.$$

These spaces have been considered by Guinand, Miller and Goldberg. Since ϕ_j is real, we have $L_{G_j} = L_{G'_j}$ and therefore $\mathscr{L} = \mathscr{H}$. Hence Watson transforms map $\mathscr{L}$ into $\mathscr{L}$.

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